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J. Math. Anal. Appl. 288 (2003) 717–721

Journal of
MATHEMATICAL
ANALYSIS AND
APPLICATIONS

www.elsevier.com/locate/jmaa

Symmetric and quantum symmetric derivatives of Lipschitz functions

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Received 23 August 2003

Submitted by M. Laczkovich

Abstract

The symmetric derivative of a real valued function f at the real number x is defined to be

$$\lim_{h\searrow 0}\frac{f(x+h)-f(x-h)}{2h}$$

when that limit exists, and if additionally $x\neq 0$, the quantum symmetric derivative is defined to be

$$\lim_{q\searrow 1}\frac{f(qx)-f(q^{-1}x)}{qx-q^{-1}x}$$

when that limit exists. An increasing function $\varphi: R^{+}\to R$ satisfying

$$\lim_{h\searrow 0}\varphi(h)/h^{1/2}=0$$

defines by $\{f\colon |f(x+h)-f(x)|\leqslant C_f\varphi(h)\}$ a class of continuous functions which we call a Lipschitz class of functions smoother than Lip $1/2$. The symmetric derivative and the quantum symmetric derivative are equivalent pointwise everywhere for functions that are in any Lipschitz class smoother than Lip $1/2$, but not necessarily for functions that are Lipschitz of order $1/2$.

Keywords: Quantum derivative; q-Derivative; Quantum symmetric derivative; Symmetric derivative; Lipschitz function

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[1] The author's research was partially supported by NSF grant DMS 9707011 and a grant from the Faculty and Development Program of the College of Liberal Arts and Sciences, DePaul University.

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2003.09.021

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The quantum derivative has been extensively studied and has proven to be useful in the study of hypergeometric series [3,4].

An increasing function $\varphi: R^+\to R$ satisfying

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defines by $\{f\colon |f(x+h)-f(x)|\leqslant C_f\varphi(h)\}$ a class of continuous functions which we call a Lipschitz class of functions smoother than $\mathrm{Lip}\,1/2$. For example, given any $\alpha>0$, $\mathrm{Lip}\,1/2+\alpha=\{f\colon |f(x+h)-f(x)|\leqslant C_f|h|^{1/2+\alpha}\}$ is a Lipschitz class of functions smoother than $\mathrm{Lip}\,1/2$.

The two generalized derivatives defined above are equivalent almost everywhere, since each is almost everywhere equivalent to ordinary differentiation [2]. We show here that this equivalence is true pointwise everywhere for functions that are in any Lipschitz class smoother than $\mathrm{Lip}\,1/2$, but not necessarily for functions that are Lipschitz of order $1/2$.

Before proving this, we will give a quick overview of the larger context. Marcinkiewicz and Zygmund founded the subject by establishing the almost everywhere equivalence of symmetric (Riemann) nth order differentiation and unsymmetric (Peano) nth order differentiation [5]. An L^p version, $1\leqslant p<\infty$, of this result took some years and effort [1,6]. Next came a quantum version of the original result [2]. The final result should be an L^p quantum version, $1\leqslant p<\infty$. But this has not been done and appears to be very difficult. The present work is part of an attack on the L^p quantum version based on trying to establish the almost everywhere equivalence of L^p quantum symmetric differentiation and L^p symmetric differentiation.

Theorem 1. *Suppose that f is in a Lipschitz class of functions smoother than* $\mathrm{Lip}\,1/2$. *Then for any fixed $x\neq 0$, the two conditions*

$$\bigl|f(x+h)-f(x+h)-a(x)2h\bigr|=o(h)\quad \text{as } h\searrow 0 \tag{1}$$

and

$$\bigl|f(qx)-f(q^{-1}x)-a(x)(q-q^{-1})x\bigr|=o(q-q^{-1})\quad \text{as } q\searrow 1 \tag{2}$$

are equivalent.

Proof. The idea is that the points qx and $q^{-1}x$ which are symmetrically placed about x in the multiplicative sense, are also “almost” placed symmetrically about x in the additive sense as well so long as q is close to 1. Another way to put this is to say that if we define h by

$$qx - q^{-1}x = 2h, \tag{3}$$

then

- of course x is at the (additive) center of $[x-h, x+h]$,
- $|[q^{-1}x, qx]| = |[x-h, x+h]|$, and
- the distances between $x+h$ and qx and between $x-h$ and $q^{-1}x$ are both so small as to be of the order of $(q-q^{-1})^2$, which is the same as being of order h^2.

Suppose Eq. (1) holds. Then defining h as above, we have

$$\begin{aligned} &f(qx) - f(q^{-1}x) - a(x)(q-q^{-1})x \\ &\quad = \{f(x+h) - f(x-h) - a(x)2h\} + \{f(qx) - f(x+h)\} \\ &\qquad + \{f(x-h) - f(q^{-1}x)\} \\ &\quad = I + II + III. \end{aligned}$$

By hypothesis (1), $I = o(h) = o(q-q^{-1})$. Next,

$$\begin{aligned} 2\{qx - (x+h)\} &= x\{2(q-1) - (q-q^{-1})\} = x\left\{2(q-1) - \frac{q^2-1}{q}\right\} \\ &= x(q-1)\left\{2 - \frac{q+1}{q}\right\} = \frac{x}{q}(q-1)^2 \end{aligned}$$

so

$$|II| \leqslant C\varphi\bigl(qx - (x+h)\bigr) = C\varphi\left(\frac{x}{2q}(q-1)^2\right) = o(q-q^{-1}),$$

and

$$\begin{aligned} 2\{q^{-1}x - (x-h)\} &= x\left\{2(q^{-1}-1) + \frac{q^2-1}{q}\right\} \\ &= \frac{x}{q}\{2(1-q) + (q-1)(q+1)\} = \frac{x}{q}(q-1)^2 \end{aligned}$$

so

$$|III| \leqslant C\varphi\left(\frac{x}{2q}(q-1)^2\right) = o(q-q^{-1}).$$

This proves that condition (1) implies condition (2). Conversely, if condition (2) holds, now use Eq. (3) to define q and write

$$\begin{aligned} &f(x+h) - f(x-h) - a(x)2h \\ &\quad = \{f(qx) - f(q^{-1}x) - a(x)(q-q^{-1})x\} + \{f(x+h) - f(qx)\} \\ &\qquad + \{f(q^{-1}x) - f(x-h)\} \\ &\quad = I' - II - III. \end{aligned}$$

Now I' is assumed to be $o(q - q^{-1}) = o(h)$ and the estimates for II and III above show these also to be $o(h)$. $\square$

The theorem is sharp in the sense that the condition that f be in a Lipschitz class of functions smoother than $\mathrm{Lip}\,1/2$ cannot be relaxed to $f \in \mathrm{Lip}\,1/2$. To see this, we will give an example of a function that is $\mathrm{Lip}\,1/2$ in a neighborhood of $x = 1$, has symmetric derivative 0 at $x = 1$, and does not have a symmetric quantum derivative at $x = 1$. It is equally easy to turn things around so that the symmetric quantum derivative exists while the symmetric derivative does not.

Example 1. Define $f(x)$, $x \in [1, 2]$, by $f(1 + 2^{-2n} + 2^{-4n}) = 1/2^{2n}$, $n = 1, 2, 3, \ldots,$ $f(1 + 2^{-n}) = 0$, $n = 0, 1, 2, 3, \ldots,$ f is linear on the intervals between adjacent above-mentioned points and $f(1) = 0$.

Extend f's domain to $[0, 2]$ by reflecting f additively symmetrically about $x = 1$,

$$f(1 - x) = f(1 + x) \quad \text{for } 0 \leqslant x \leqslant 1. \tag{4}$$

Condition (4) immediately guarantees that f has symmetric derivative 0 at $x = 1$.

To see that f does not have a symmetric quantum derivative at $x = 1$, suppose that $q = 1/(1 - 2^{-2n})$. Then $f(q^{-1}) = 0$, while

$$1 + 2^{-2n} + 2^{-4n} < q = 1 + 2^{-2n} + 2^{-4n}\frac{1}{1 - 2^{-2n}} \leqslant 1 + 2^{-2n} + 2^{-4n+1},$$

so that

$$\begin{aligned} f(q) > f(1 + 2^{-2n} + 2^{-4n+1}) &= \frac{1}{2^{2n}}\frac{(1 + 2^{-2n} + 2^{-4n+1}) - (1 + 2^{-2n+1})}{(1 + 2^{-2n} + 2^{-4n}) - (1 + 2^{-2n+1})} \\ &= \frac{1}{2^{2n}}\frac{-2^{-2n} + 2^{-4n+1}}{-2^{-2n} + 2^{-4n}} \cong \frac{1}{2^{2n}}, \end{aligned}$$

since on $[1 + 2^{-2n} + 2^{-4n}, 1 + 2^{-2n+1}]$,

$$f(x) = \frac{1}{2^{2n}}\frac{x - (1 + 2^{-2n+1})}{(1 + 2^{-2n} + 2^{-4n}) - (1 + 2^{-2n+1})}$$

has negative slope. Also

$$q - q^{-1} = \frac{1}{1 - 2^{-2n}} - (1 - 2^{-2n}) \cong \frac{2}{2^{2n}}$$

so that

$$\limsup_{q \searrow 1}\frac{f(q) - f(q^{-1})}{q - q^{-1}} \geqslant \lim_{n\to\infty}\frac{1/2^{2n}}{2/2^{2n}} = \frac{1}{2}.$$

On the other hand, if $q = 1/(1 - 2^{-2n+1})$, again $q^{-1} = 1 - 2^{-2n+1}$ so $f(q^{-1}) = 0$. Now, however, $q = 1 + 2^{-2n+1}/(1 - 2^{-2n+1})$ is in the interval $[1 + 2^{-2n+1}, 1 + 2^{-2n+2}]$ so $f(q) = 0$ also. Thus

$$\liminf_{q \searrow 1}\frac{f(q) - f(q^{-1})}{q - q^{-1}} = 0,$$

and f does not have a symmetric quantum derivative at $x = 1$.

It remains to show that $f \in \mathrm{Lip}\,1/2$, with $\mathrm{Lip}\,1/2$ constant 1. Define an interval function

$$S(a,b) = S\big([a,b]\big) := \frac{|f(b)-f(a)|}{\sqrt{b-a}},$$

where a and b are the endpoints of the interval.

The graph of f is made up of two sequences of triangles that converge toward $x=1$. We must show that for all $x<y$ in $[0,2]$, $S(x,y) \leqslant 1$. The symmetry of the situation allows us to assume that y is on the base of a triangle that lies to the right of 1 and also that if x lies to the left of that triangle, then $f(x) < f(y)$. To follow this proof draw one of these triangles by connecting the points $(a,0)$, $(b, f(b))$ and $(c,0)$, where $a<b<c$. Let $L = \overline{(a,0)(b, f(b))}$ and $R = \overline{(b, f(b))(c,0)}$. Since the triangle lies to the right of $x=1$, then for some integer $n \geqslant 1$, $a = 1+2^{-2n}$, $b = 1+2^{-2n}+2^{-4n}$, and $c = 1+2^{-2n+1}$.

Case 1: $(x,y)=(a,b)$. We have $S(a,b)=1$ since

$$f(1+2^{-2n}+2^{-4n}) - f(1+2^{-2n}) = 2^{-2n} = \cdot\sqrt{(1+2^{-2n}+2^{-4n})-(1+2^{-2n})}.$$

Case 2: $(x,y)=(b,c)$. Note that $c-b = 2^{-2n+1}-(2^{-2n}+2^{-4n}) > 2^{-4n} = b-a$. So $S(b,c) < S(a,b)$ and Case 1 applies.

Case 3: $[x,y] \subset [a,b]$ or $[x,y] \subset [b,c]$. It suffices to note that if $[x,y] \subset [\alpha,\beta]$ and if f is linear on $[\alpha,\beta]$, then

$$\begin{aligned} S(x,y) &= \frac{|f(y)-f(x)|}{|y-x|}|y-x|^{1/2} = \frac{|f(\beta)-f(\alpha)|}{|\beta-\alpha|}|y-x|^{1/2} \\ &\leqslant \frac{|f(\beta)-f(\alpha)|}{|\beta-\alpha|}|\beta-\alpha|^{1/2} = S(\alpha,\beta). \end{aligned}$$

Case 4: $a \leqslant x < b < y \leqslant c$. Visualize the segment $\Delta = \overline{(y, f(y))(x, f(x))}$ as a solid rod. If $f(y) > f(x)$, slide the rod to the left, keeping the left end on L and the right end on M and stop when either (i) the right end reaches $(b, f(b))$ or (ii) the right end reaches $(a,0)$. Throughout this process $S(x,y)$ increases. If (i) happens, we are back to Case 3. If (ii) happens, it is now clear that $S(x,y) < S(a,b)$. If $f(x) > f(y)$, slide the rod to the right and reason similarly.

Case 5: $x < a < y \leqslant c$. By hypothesis, Δ has positive slope. Then $S(x,y) < S(a,y)$. Now one of the earlier cases must apply.

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